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Let $\underset{\sim}{X}(1),\ldots,\underset{\sim}{X}(n)$ be $p\times 1$ random vectors with mean zero. Put $S = \frac{1}{n}\sum_{t=1}^{n} \underset{\sim}{X}(t)\underset{\sim}{X}(t)'$. When $\underset{\sim}{X}(t)$, $(t=1,\ldots,n)$ are distributed as i.i.d. $N(\underset{\sim}{0},\Sigma)$, (i.e., usual multivariate analysis), many authors have investigated the asymptotic expansions for the distributions of various functions of the eigenvalues of S. In this paper we will extend the above results ot the case when $\{\underset{\sim}{X}(t)\}$ is a Guassian stationary process. Also we shall derive the asymptotic expansions for certain

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ASYMPTOTIC DISTRIBUTIONS OF FUNCTIONS OF EIGENVALUES OF THE SAMPLE COVARIANCE MATRIX AND CANONICAL CORRELATION MATRIX IN MULTIVARIATE TIME SERIES*

M. Taniguchi**

P.R. Krishnaiah

Center for Multivariate Analysis
University of Pittsburgh

Center for Multivariate Analysis

University of Pittsburgh

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M. Taniguchi**

P.R. Krishnaiah

Center for Multivariate Analysis
University of Pittsburgh

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Center for Multivariate Analysis
Fifth Floor, Thackeray Hall
University of Pittsburgh
Pittsburgh, PA 15260

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**The work of this author was done at the Center for Multivariate Analysis. His permanent address is the Department of Mathematics, Hiroshima University, Hiroshima Japan.

ABSTRACT

Let $\underset{\sim}{X}(1),\ldots,\underset{\sim}{X}(n)$ be px1 random vectors with mean zero. Put $S = \frac{1}{n}\sum_{t=1}^{n} \underset{\sim}{X}(t)\underset{\sim}{X}(t)'$. When $\underset{\sim}{X}(t),(t=1,\ldots,n)$ are distributed as i.i.d. $N(0,\underset{\sim}{\Sigma})$, (i.e., usual multivariate analysis), many authors have investigated the asymptotic expansions for the distributions of various functions of the eigen values of S. In this paper we will extend the above results to the case when $\{\underset{\sim}{X}(t)\}$ is a Gaussian stationary process. Also we shall derive the asymptotic expansions for certain functions of the sample canonical correlations in multivariate time series. Applications of some of the results in signal processing are also discussed.

1. INTRODUCTION

Recently several authors have introduced some multivariate methods to multivariate time series analysis. Using the orthogonality and asymptotic normality of the "Fourier components" of time series, Priestley, Rao and Tong (1973), Brillinger (1975) and Krishnaiah (1976) discussed the principal component analysis in the frequency domain, and investigated some asymptotic properties of sample principal component roots and test statistics. Similarly Hannan (1970), Brillinger (1975) and Krishnaiah (1976) discussed the canonical correlation analysis in the frequency domain, and Brillinger gave the limiting distributions of estimates of canonical correlation coefficients. Also Robinson (1973) gave an approach of canonical correlation analysis for the covariance matrix of multivariate time series.

In this paper we shall discuss the asymptotic distributions of eigenvalues of sample covariance matrices of multivariate time series since the eigenvalues play a fundamental role in multivariate problems. In Section 2, we shall give the limiting distribution of eigenvalues of sample covariance matrices for non-Gaussian linear vector processes. Furthermore, in Section 3, we shall derive the asymptotic expansions of certain functions of eigenvalues of covariance matrix for multivariate Gaussian stationary processes, and discuss their applications for time series principal component analysis.

In Section 4 ~~we shall~~ give the asymptotic expansions of certain functions of canonical correlation matrix for multivariate Gaussian stationary processes, and discuss some asymptotic properties of a test statistic for canonical correlations.

2. CENTRAL LIMIT THEOREM FOR EIGENVALUES OF COVARIANCE MATRICES

In this section we give the central limit theorem for eigenvalues of sample covariance matrices. In the sequel, the set of all integers is denoted by J and the Kronecker's delta is denoted by $\delta(m,n)$.

Let $\{\underset{\sim}{X}(t) : t \in J\}$ be a vector-valued linear process generated as

$$\underset{\sim}{X}(t) = \sum_{j=0}^{\infty} G(j)\, \underset{\sim}{e}(t-j), \quad t \in J,$$

where the $\underset{\sim}{X}(t)$'s have p components and the $\underset{\sim}{e}(n)$'s are q-vectors such that $E\{\underset{\sim}{e}(n)\} = \underset{\sim}{0}$ and $E\{\underset{\sim}{e}(m), \underset{\sim}{e}(n)'\} = \delta(m,n)K$, with K a nonsingular $q \times q$ matrix; the $G(j)$'s are $p \times q$ matrices; and the components of $\underset{\sim}{X}$, $\underset{\sim}{e}$ and G are all real.

If $\sum_{j=0}^{\infty} \mathrm{tr}G(j)KG(j)' < \infty$ (this condition is assumed throughout in this section), the process $\{\underset{\sim}{X}(t)\}$ is a second-order stationary process and has a spectral density matrix $\mathfrak{f}(w)$ which is representable as

$$\mathfrak{f}(w) = \frac{1}{2\pi} k(w)K\, k(w)^*, \quad -\pi \le w \le \pi,$$

where

$$k(w) = \sum_{j=0}^{\infty} G(j)\, e^{iwj}.$$

Suppose that observed stretch $\{\underset{\sim}{X}(1),\ldots,\underset{\sim}{X}(n)\}$ of $\{\underset{\sim}{X}(t)\}$ is available. Define $C(0) = \frac{1}{n}\sum_{t=1}^{n} \underset{\sim}{X}(t)\underset{\sim}{X}(t)'$, and denote the (α,β) components of $C(0)$, $\mathfrak{f}(w)$, $k(w)$ and K by $C(\alpha,\beta)$, $\mathfrak{f}_{\alpha\beta}(w)$, $k_{\alpha\beta}(w)$ and $K_{\alpha\beta}$, and denote the αth component of

$\underset{\sim}{X}(t)$ and $\underset{\sim}{e}(t)$ by $X_\alpha(t)$ and $e_\alpha(t)$, respectively. Assuming that $\{\underset{\sim}{e}(t)\}$ is fourth-order stationary, let $Q^e_{\alpha_1 \ldots \alpha_4}(t_1,t_2,t_3)$ be the joint fourth cumulant of $e_{\alpha_1}(t)$, $e_{\alpha_2}(t+t_1)$, $e_{\alpha_3}(t+t_2)$, $e_{\alpha_4}(t+t_3)$ and assume that

$$\sum_{t_1,t_2,t_3=-\infty}^{\infty} |Q^e_{\alpha_1 \ldots \alpha_4}(t_1,t_2,t_3)| < \infty, \quad (1 \leq \alpha_1,\ldots,\alpha_4 \leq q);$$

then the process $\{\underset{\sim}{e}(t)\}$ has the fourth-order spectral density $\tilde{Q}^e_{\alpha_1 \ldots \alpha_4}(w_1,w_2,w_3)$ such that

$$\tilde{Q}^e_{\alpha_1 \ldots \alpha_4}(w_1,w_2,w_3) = \frac{1}{(2\pi)^3} \sum_{t_1,t_2,t_3=-\infty}^{\infty} \exp\{-i(w_1t_1+w_2t_2+w_3t_3)\}$$

$$\times\, Q^e_{\alpha_1 \ldots \alpha_4}(t_1,t_2,t_3).$$

Similarly we can define $Q^x_{\alpha_1 \ldots \alpha_4}(t_1,t_2,t_3)$ and $\tilde{Q}^x_{\alpha_1 \ldots \alpha_4}(w_1,w_2,w_3)$, respectively, the fourth-order cumulant and spectral density of the process $\{\underset{\sim}{X}(t)\}$, $1 \leq \alpha_1,\ldots,\alpha_4 \leq p$. Denote by $\beta(t)$ the σ-field generated by $\{e(n);\ n \leq t\}$. We set down the following assumptions.

<u>Assumption 1</u>. For each β_1,β_2 and s

$$\mathrm{Var}[E\{e_{\beta_1}(t)e_{\beta_2}(t+s)|\beta(t-\tau)\} - \delta(s,0)K_{\beta_1\beta_2}]$$

$$= O(\tau^{-2-\varepsilon}),\ \varepsilon > 0,$$

uniformly in t.

Assumption 2.

$$E\left|E\{e_{\beta_1}(t_1)e_{\beta_2}(t_2)e_{\beta_3}(t_3)e_{\beta_4}(t_4)\,|\,\beta(t_1-\tau)\}\right.$$

$$\left. - E\{e_{\beta_1}(t_1)e_{\beta_2}(t_2)e_{\beta_3}(t_3)e_{\beta_4}(t_4)\}\right| = O(\tau^{-1-\eta}),$$

uniformly in t_1, where $t_1 \leq t_2 \leq t_3 \leq t_4$ and $\eta > 0$.

Assumption 3.

The spectral densities $f_{\beta\beta}(w)$ $(\beta = 1,\ldots,p)$ are square-integrable.

Assumption 4.

$$\sum_{t_1,t_2,t_3=-\infty}^{\infty} |Q^e_{\alpha_1\ldots\alpha_4}(t_1,t_2,t_3)| < \infty.$$

Now define $\Gamma(0) = E\{C(0)\}$, and denote the (α,β) component of $\Gamma(0)$ by $\Gamma(\alpha,\beta)$. The following lemma is due to Hosoya and Taniguchi (1982).

Lemma 1.

Under Assumptions 1-4,

$$\sqrt{n}\,\{C(\alpha_1,\alpha_2) - \Gamma(\alpha_1,\alpha_2)\},\ (\alpha_1,\alpha_2 = 1,\ldots,p)$$

have a joint asymptotic normal distribution whose mean is zero and the asymptotic covariance between $\sqrt{n}\,\{C(\alpha_1,\alpha_2) - \Gamma(\alpha_1,\alpha_2)\}$ and $\sqrt{n}\,\{C(\alpha_3,\alpha_4) - \Gamma(\alpha_3,\alpha_4)\}$ is given as

$$2\pi \int_{-\pi}^{\pi} \{ f_{\alpha_1\alpha_3}(w)\overline{f_{\alpha_2\alpha_4}(w)} + f_{\alpha_1\alpha_4}(w)\overline{f_{\alpha_2\alpha_3}(w)} \} \, dw$$

$$+ 2\pi \sum_{\beta_1,\dots,\beta_4=1}^{p} \iint_{-\pi}^{\pi} k_{\alpha_1\beta_1}(w_1)k_{\alpha_2\beta_2}(-w_1)k_{\alpha_3\beta_3}(w_2)k_{\alpha_4\beta_4}(-w_2)$$

$$\times \tilde{Q}^{e}_{\beta_1\dots\beta_4}(w_1,-w_2,w_2)dw_1dw_2. \qquad \square$$

Let $\ell_1 \geq \dots \geq \ell_p$ be the eigenvalues of $C(0)$. Then we can express $C(0) = BLB'$, where $L = \mathrm{diag}(\ell_1,\dots,\ell_p)$, B is the orthogonal matrix. Suppose that the eigenvalues of $\Gamma(0)$ satisfy $\lambda_1 > \dots > \lambda_p$, and that $\Gamma(0) = \Psi\Lambda\Psi'$ where $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_p)$ and $\Psi = (\underset{\sim}{\beta}_1,\dots,\underset{\sim}{\beta}_p)$ is the orthogonal matrix. We assume that $\lambda_1,\dots,\lambda_p$ have the following spectral representations

$$\lambda_j = \int_{-\pi}^{\pi} \tilde{f}_{jj}(w)dw, \quad (j = 1,\dots,p), \tag{2.1}$$

where $\tilde{f}_{jj}(w) = \frac{K_{jj}}{2\pi} |\tilde{k}_{jj}(w)|^2$, and $\tilde{k}_{jj}(w) = \sum_{\ell=0}^{\infty} \tilde{G}_{jj}(\ell)e^{i\ell w}$ with $\sum_{\ell=0}^{\infty} |\tilde{G}_{jj}(\ell)|^2 < \infty$.

Let $T = \Psi'C(0)\Psi$, and put $U = \sqrt{n}(T-\Lambda)$. Then, denoting the (i.j) component of U by u_{ij}, from Lemma 1, $\{u_{ij}\}$ have a joint asymptotic normal distribution whose mean is zero and the asymptotic covariance between u_{ij} and $u_{k\ell}$ is given as

$$\mu(ij,k\ell) = 2\pi \int_{-\pi}^{\pi} \{ g_{ik}(w)\overline{g_{j\ell}(w)} + g_{i\ell}(w)\overline{g_{jk}(w)} \} \, dw$$

$$+ 2\pi \iint_{-\pi}^{\pi} \tilde{k}_{ii}(w_1)\tilde{k}_{jj}(-w_1)\tilde{k}_{kk}(w_2)\tilde{k}_{\ell\ell}(-w_2)\tilde{Q}^{e}_{ijk\ell}(w_1,-w_2,w_2)$$

$$\times dw_1dw_2, \tag{2.2}$$

where $g_{ij}(w) = \tilde{f}_{ii}(w)\delta(i,j)$. Then we get the following theorem.

Theorem 1. Under Assumptions 1-4, the joint distribution of $D = \sqrt{n}(L-\Lambda)$ and $G = \sqrt{n}(B-\Psi)$ tends to that of $\tilde{D} = \text{diag}(\tilde{u}_{11},\ldots,\tilde{u}_{pp})$ and $\tilde{G} = \Psi\tilde{W}$, where $\tilde{W} = \{\tilde{w}_{ij}\}$, with $\tilde{w}_{ii} = 0$, $\tilde{w}_{ij} = \dfrac{\tilde{u}_{ij}}{\lambda_j-\lambda_i}$, $i \neq j$ and $\{\tilde{u}_{ij}\}$ is distributed as a multivariate normal distribution whose mean is zero vector and the covariance between $\tilde{u}_{ij}$ and $\tilde{u}_{k\ell}$ is given as $\mu(ij,k\ell)$ defined in (2.2). In particular if the process is Gaussian, then the limiting distribution of D and G = $(\underset{\sim}{h}_1,\ldots,\underset{\sim}{h}_p)$ is normal with D and G independent and the diagonal elements of D are independent. The diagonal element d_i of D has the limiting distribution

$$N(0,\ 4\pi \int_{-\pi}^{\pi} g_{ii}(w)^2 dw).$$

The covariance matrix of the ith eigenvector is

$$\text{Var}\{\underset{\sim}{h}_i\} = \sum_{\substack{k=1\\k\neq i}}^{p} \frac{2\pi}{(\lambda_i-\lambda_k)^2} \int_{-\pi}^{\pi} g_{ii}(w)g_{kk}(w)dw\cdot \underset{\sim}{\beta}_k \underset{\sim}{\beta}_k',$$

and the covariance matrix of $\underset{\sim}{h}_i$ and $\underset{\sim}{h}_j$ in the limiting distribution is

$$\text{Cov}\{\underset{\sim}{h}_i,\underset{\sim}{h}_j\} = -\frac{2\pi}{(\lambda_i-\lambda_j)^2} \int_{-\pi}^{\pi} g_{ii}(w)g_{jj}(w)dw\cdot \underset{\sim}{\beta}_j \underset{\sim}{\beta}_i'.$$

[Proof] Let $T = YLY'$, where Y is orthogonal. Put $W = \sqrt{n}(Y-I) = \{w_{ij}\}$. Using the same arguments as Anderson (1984, p. 541), and substituting $T = \Lambda + \frac{1}{\sqrt{n}} U$, $Y = I + \frac{1}{\sqrt{n}} W$ and $L = \Lambda + \frac{1}{\sqrt{n}} D$ to the equations $T = YLY'$ and

$I = YY'$, we get $d_i = u_{ii}$, $(i = 1,\dots,p)$, $w_{ii} = 0$, $w_{ij} = \frac{u_{ij}}{\lambda_j - \lambda_i}$, $i \neq j$, $i\ j = 1,\dots,p$. Noting $\sqrt{n}\,(B-\Psi) = \Psi W$ and Lemma 1, we get the desired results. □

Remark. For non-Gaussian case, the distribution of D and G are not always asymptotically independent.

Now we confine ourselves to the case when $\{\underset{\sim}{X}(t)\}$ is Gaussian and consider the asymptotic distribution of the q smallest roots of $C(0)$ when the q smallest population roots are equal. Let

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \theta I_q \end{pmatrix},$$

where the diagonal elements of the diagonal matrix Λ_1 are different and larger than θ (>0). We assume that the eigenvalues $\lambda_1,\dots,\lambda_p$ $(\lambda_1 > \dots > \lambda_{p-q} > \lambda_{p-q+1} = \dots = \lambda_p = \theta)$ have the following spectral representations given in (2.1);

$$\lambda_j = \int_{-\pi}^{\pi} \tilde{f}_{jj}(w)dw, \tag{2.3}$$

with $\tilde{f}_{p-q+1,p-q+1}(w) = \dots = \tilde{f}_{p,p}(w) = \frac{\theta}{2\pi}$, (i.e., uniform spectra).

As a criterion to test the null hypothesis

$$H : \lambda_1 > \dots > \lambda_{p-q} > \lambda_{p-q+1} = \dots = \lambda_p = \theta, \tag{2.4}$$

where λ_j $(j = 1,\dots,p)$ have the spectral representations (2.3), we use the following criterion

$$L = -n \log \frac{\prod_{i=p-q+1}^{p} \ell_i}{\left(\frac{1}{q} \sum_{i=p-q+1}^{p} \ell_i\right)^q} . \tag{2.5}$$

In the usual multivariate analysis this is known as the likelihood ratio criterion. Using the similar arguments in Anderson (1984, p. 475-6), we can see

$$L = \frac{1}{\theta^2} [\sum_{\substack{i<j \\ p-q+1 \leq i \\ j \leq p}} u_{ij}^2 + \tfrac{1}{2}\{ \sum_{i=p-q+1}^{p} u_{ii}^2$$

$$- \frac{1}{q} (\sum_{i=p-q+1}^{p} u_{ii})^2 \}] + \text{higher order terms.} \tag{2.6}$$

Since we assumed (2.3), we can see that

$$u_{ii} (i = p-q+1,\dots,p) \text{ and } u_{ij} (i \neq j,\ i \cdot j = p-q+1,\dots,p)$$

are asymptotically distributed as i.i.d. $N(0, 2\theta^2)$ and i.i.d. $N(0, \theta^2)$, respectively. By (2.6) we have,

Theorem 2. Under the null hypothesis H, the limiting distribution of the test statistic L tends to the x^2-distribution with $\frac{1}{2}(q+2)(q-1)$ degrees of freedom.

3. ASYMPTOTIC EXPANSIONS OF CERTAIN FUNCTIONS OF EIGENVALUES OF COVARIANCE MATRIX

In this section we derive the asymptotic expansions of certain functions of eigenvalues of covariance matrix in multivariate time series. Here we assume that $\{\underset{\sim}{X}(t) = (X_1(t),\ldots,X_p(t))' : t\varepsilon J\}$ is a px1-vector-valued Gaussian stationary process with zero mean and covariance matrices $\Gamma(j) = E\{\underset{\sim}{X}(t)\underset{\sim}{X}(t+j)'\}$.

<u>Assumption 5</u>. The covariance matrices satisfy

$$\sum_{u=-\infty}^{\infty} |u| \, \|\Gamma(u)\| < \infty, \tag{3.1}$$

where $\|(\Gamma(u)\|$ is the square root of the maximum eigenvalue of $\Gamma(u)\Gamma(u)'$.

Then the spectral density matrix of $\{\underset{\sim}{X}(t)\}$ is given by

$$\mathfrak{f}(w) = \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} \Gamma(u) e^{-iuw}.$$

Let $\ell_1 \geq \ldots \geq \ell_p$ be the eigenvalues of the matrix $C(0) = \frac{1}{n} \sum_{t=1}^{n} \underset{\sim}{X}(t)\underset{\sim}{X}(t)'$ and let $\lambda_1 \geq \ldots \geq \lambda_p$ be the eigenvalues of $\Gamma(0)$ such that

$$\lambda_{q_1+\ldots+q_{\alpha-1}+1} = \ldots = \lambda_{q_1+\ldots+q_\alpha} = \theta_\alpha \tag{3.2}$$

for $\alpha = 1,2,\ldots,r$, $q_1+\ldots+q_r = p$, and $q_o = 0$. Also, let $T_j(\ell_1,\ldots,\ell_p)$, $(j = 1,\ldots,k)$, be an analytic function of $\ell_1,\ldots,\ell_p$ about $\lambda_1,\ldots,\lambda_p$. In addition, let

$$a_{j\alpha} = \left.\frac{\partial T_j(\ell_1,\ldots,\ell_p)}{\partial \ell_g}\right|_{\underset{\sim}{\ell} = \underset{\sim}{\lambda}},$$

$$a_{j\alpha\beta} = \left.\frac{\partial^2 T_j(\ell_1,\ldots,\ell_p)}{\partial \ell_g \, \partial \ell_h}\right|_{\underset{\sim}{\ell} = \underset{\sim}{\lambda}},$$

$$a_{j\alpha\beta\gamma} = \left.\frac{\partial^3 T_j(\ell_1,\ldots,\ell_p)}{\partial \ell_g\, \partial \ell_h\, \partial \ell_t}\right|_{\underset{\sim}{\ell} = \underset{\sim}{\lambda}},$$

for $g\varepsilon J_\alpha$, $h\varepsilon J_\beta$, $t\varepsilon J_\gamma$, $\underset{\sim}{\lambda}' = (\lambda_1,\ldots,\lambda_p)$ and $\underset{\sim}{\ell}' = (\ell_1,\ldots,\ell_p)$ where J_α denotes the set of integers $q_1+\ldots+q_{\alpha-1}+1,\ \ldots,\ q_1+\ldots+q_\alpha$ for $\alpha = 1,2,\ldots,r$. In this section we are interested in obtaining the asymptotic joint distribution of

$$\sqrt{n}\{T_j(\ell_1,\ldots,\ell_p) - T_j(\lambda_1,\ldots,\lambda_p)\},\ (j = 1,\ldots,k).$$

Expanding $T_j(\ell_1,\ldots,\ell_p)$ as Taylor series, we obtain

$$T_j(\ell_1,\ldots,\ell_p) = T_j(\lambda_1,\ldots,\lambda_p) + \sum_{\alpha=1}^{r} a_{j\alpha} \sum_{g\varepsilon J_\alpha} (\ell_g-\theta_\alpha)$$

$$+ \frac{1}{2} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} a_{j\alpha\beta} \sum_{g\varepsilon J_\alpha} \sum_{h\varepsilon J_\beta} (\ell_g-\theta_\alpha)(\ell_h-\theta_\beta)$$

$$+ \frac{1}{6} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{\gamma=1}^{r} a_{j\alpha\beta\gamma} \sum_{g\varepsilon J_\alpha} \sum_{h\varepsilon J_\beta} \sum_{t\varepsilon J_\gamma} (\ell_g-\theta_\alpha)(\ell_h-\theta_\beta)(\ell_t-\theta_\gamma)$$

+ higher order terms.

Now, let $L_j = \sqrt{n}\{T_j(\ell_1,\ldots,\ell_p) - T_j(\lambda_1,\ldots,\lambda_p)\}$ for $j = 1,2,\ldots,k$.

Then

$$L_j = \sqrt{n} \sum_{\alpha=1}^{r} a_{j\alpha}\, \mathrm{tr}(W_\alpha-\theta_\alpha I) + \frac{\sqrt{n}}{2} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} a_{j\alpha\beta}$$

$$\times\ \mathrm{tr}(W_\alpha-\theta_\alpha I)\mathrm{tr}(W_\beta-\theta_\beta I)$$

$$+ \frac{\sqrt{n}}{6} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{\gamma=1}^{r} a_{j\alpha\beta\gamma}\, \mathrm{tr}(W_\alpha-\theta_\alpha I)\mathrm{tr}(W_\beta-\theta_\beta I)\mathrm{tr}(W_\gamma-\theta_\gamma I)$$

+ higher order terms,

where W_α is a random matrix with eigenvalues ℓ_g, $g\varepsilon J_\alpha$, and I is an identity matrix. If we assume that W_α is of the form;

$$W_\alpha = \theta_\alpha I + \frac{1}{\sqrt{n}} W_\alpha^{(1)} + \frac{1}{n} W_\alpha^{(2)} + \dots ,$$

then L_j is of the following form

$$L_j = \sum_{\alpha=1}^{r} a_{j\alpha} \operatorname{tr} W_\alpha^{(1)} + \frac{1}{\sqrt{n}}\{ \sum_{\alpha=1}^{r} a_{j\alpha} \operatorname{tr} W_\alpha^{(2)}$$

$$+ \frac{1}{2} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} a_{j\alpha\beta} \operatorname{tr} W_\alpha^{(1)} \operatorname{tr} W_\beta^{(1)} \}$$

+ higher order terms.

The following lemma is due to Fujikoshi (1977).

Lemma 2. Let T be a square random matrix and $d_1 \geq \dots \geq d_p$ be the eigenvalues of T. Also, let $\lambda_1 \geq \dots \geq \lambda_p$ be the corresponding population eigenvalue such that

$$\lambda_1 = \dots = \lambda_{q_1} = \theta_1,$$

$$\lambda_{q_1+1} = \dots = \lambda_{q_1+q_2} = \theta_2$$

$$\vdots$$

$$\lambda_{q_1+\dots+q_{r-1}} = \dots = \lambda_p = \theta_r$$

In addition, we assume that T can be expressed as

$$T = \Lambda + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \dots ,$$

where $\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_p)$ and $\varepsilon > 0$ is very small. Then the eigenvalues d_j ($j \varepsilon J_\alpha$) are the eigenvalues of

$$Z_\alpha = \theta_\alpha I_{q_\alpha} + \varepsilon Z_\alpha^{(1)} + \varepsilon^2 Z_\alpha^{(2)} + \dots ,$$

where

$$Z_\alpha^{(1)} = U_{\alpha\alpha}^{(1)}, \; Z_\alpha^{(2)} = U_{\alpha\alpha}^{(2)} + \sum_{\beta\neq\alpha}^{r} (\theta_\alpha - \theta_\beta)^{-1} U_{\alpha\beta}^{(1)} U_{\beta\alpha}^{(1)} ,$$

$$U^{(i)} = \begin{pmatrix} U_{11}^{(i)} & U_{12}^{(i)} & \cdots & U_{1r}^{(i)} \\ U_{21}^{(i)} & U_{22}^{(i)} & \cdots & U_{2r}^{(i)} \\ \vdots & & & \vdots \\ U_{r1}^{(i)} & U_{r2}^{(i)} & \cdots & U_{rr}^{(i)} \end{pmatrix},$$

and $U_{\alpha\beta}^{(i)}$ is of order $q_\alpha \times q_\beta$. □

Applying this lemma to our normalized covariance matrix U defined in the previous section we have

$$L_j = \sum_{\alpha=1}^{r} a_{j\alpha} \mathrm{tr} U_{\alpha\alpha} + \frac{1}{\sqrt{n}} \{ \sum_{\alpha=1} a_{j\alpha} \mathrm{tr} \sum_{\beta\neq\alpha} (\theta_\alpha - \theta_\beta)^{-1} U_{\alpha\beta} U_{\beta\alpha}$$

$$+ \tfrac{1}{2} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} a_{j\alpha\beta} \mathrm{tr} U_{\alpha\alpha} \mathrm{tr} U_{\beta\beta} \} + \text{higher order terms.} \tag{3.3}$$

Here we also assume that the eigenvalues have the spectral representations in (2.1). Applying Lemma 1 for Gaussian case it is not difficult to show that

$$E(L_j) = \frac{1}{\sqrt{n}} \{ \sum_{\alpha=1}^{r} \sum_{\beta\neq\alpha} \sum_{i\varepsilon J_\alpha} \sum_{k\varepsilon J_\beta} a_{j\alpha} (\theta_\alpha - \theta_\beta)^{-1}$$

$$\times 2\Pi \int_{-\pi}^{\pi} \tilde{f}_{ii}(w) \tilde{f}_{kk}(w) dw$$

$$+ \tfrac{1}{2} \sum_{\alpha=1}^{r} \sum_{i\varepsilon J_\alpha} a_{j\alpha\alpha} 4\pi \int_{-\pi}^{\pi} \tilde{f}_{ii}(w)^2 dw \} + O(n^{-1})$$

$$= c_j/\sqrt{n} + O(n^{-1}), \text{ (say).} \tag{3.4}$$

$$\mathrm{cum}\{L_j, L_m\} = \mathrm{cum}\{ \sum_{\alpha=1}^{r} \sum_{k\varepsilon J_\alpha} a_{j\alpha} U_{kk}, \sum_{\alpha'=1}^{r} \sum_{k'\varepsilon J_{\alpha'}} a_{m\alpha'} U_{k'k'} \}$$

$$+ 0(n^{-1})$$

$$= \sum_{\alpha=1} \sum_{k\varepsilon J_\alpha} a_{j\alpha} a_{m\alpha} 4\Pi \int_{-\pi}^{\pi} \tilde{\delta}_{kk}(w)^2 dw + 0(n^{-1})$$

$$= c_{jm} + 0(n^{-1}), \text{ (say).} \tag{3.5}$$

Now define

$$Z_n(i,j) = \frac{1}{\sqrt{n}} [\sum_{t=1}^{n} \{X_i(t)X_j(t) - \Gamma(i,j)\}],$$

and

$$\underset{\sim}{X}^{(i)} = (X_i(1),\ldots,Z_i(n))'.$$

Then we can express as

$$\underset{\sim}{X}^{(i)} = \sum_i^{\frac{1}{2}} \underset{\sim}{Y}^{(i)}$$

where $\sum_i$ is the covariance matrix of $\underset{\sim}{X}^{(i)}$, and $\underset{\sim}{Y}^{(i)}$ is a random vector distributed as $N(0_n, I_n)$. Using the fundamental properties of cumculants (see Brillinger (1975)) we have

$$\begin{aligned} &\text{cum}\{Z_n(i_1,j_1),\ Z_n(i_2,j_2),\ Z_n(i_3,j_3)\} \\ &= \frac{1}{n\sqrt{n}} \text{cum}\{\underset{\sim}{Y}^{(i_1)'} \sum_{i_1}^{\frac{1}{2}} \sum_{j_1}^{\frac{1}{2}} \underset{\sim}{Y}^{(j_1)},\ \underset{\sim}{Y}^{(i_2)'} \sum_{i_2}^{\frac{1}{2}} \sum_{j_2}^{\frac{1}{2}} \underset{\sim}{Y}^{(j_2)}, \\ &\qquad Y^{(i_3)'} \sum_{i_3}^{\frac{1}{2}} \sum_{j_3}^{\frac{1}{2}} \underset{\sim}{Y}^{(j_3)}\} \\ &= \frac{1}{n\sqrt{n}} \sum^{*} \text{tr} \sum_{\ell_1}^{\frac{1}{2}} \sum_{\ell_2}^{\frac{1}{2}} \sum_{\ell_3}^{\frac{1}{2}} \sum_{\ell_4}^{\frac{1}{2}} \sum_{\ell_5}^{\frac{1}{2}} \sum_{\ell_6}^{\frac{1}{2}} \\ &= \frac{1}{n\sqrt{n}} \sum^{*} \text{tr}\ \Gamma_{\ell_1\ell_2}(n\times n)\Gamma_{\ell_3\ell_4}(n\times n)\Gamma_{\ell_5\ell_6}(n\times n), \end{aligned} \tag{3.6}$$

where Σ^* is the sum over all two dimensional indecomposable partitions of $\begin{pmatrix} i_1,j_1 \\ i_2,j_2 \\ i_3,j_3 \end{pmatrix}$, and $\Gamma_{\ell_1\ell_2}(n\times n) = \sum_{\ell_1}^{\frac{1}{2}} \sum_{\ell_2}^{\frac{1}{2}}$. To evaluate (3.6), after a slight

modification of Theorem 1 in Taniguchi (1983) we get the following lemma.

Lemma 3. Let $\underset{\sim}{X}(t) = (X_1(t),\ldots,X_p(t))'$ be a Gaussian stationary process which satisfies Assumption 5, with the spectral density matrix $\mathfrak{f}(w) = \{\mathfrak{f}_{ij}(w)\}$ and mean zero. Then

$$\frac{1}{n}\,\mathrm{tr}\,\Gamma_{i_1j_1}(n\mathrm{x}n)\Gamma_{i_2j_2}(n\mathrm{x}n)\ \ldots\ \Gamma_{i_sj_s}(n\mathrm{x}n)$$

$$= (2\pi)^{s-1}\int_{-\pi}^{\pi}\mathfrak{f}_{i_1j_1}(w)\mathfrak{f}_{i_2j_2}(w)\cdots\mathfrak{f}_{i_sj_s}(w)dw + O(n^{-1}). \qquad (3.7)$$

Using this lemma for (3.6) and noting fundamental properties of cumulants it is not difficult to show

$$\mathrm{cum}\{L_j,L_m,L_s\} = \frac{32\pi^2}{\sqrt{n}}\sum_{\alpha=1}^{r}\sum_{k\in J_\alpha} a_{j\alpha}a_{m\alpha}a_{s\alpha}\int_{-\pi}^{\pi}\tilde{\mathfrak{f}}_{kk}(w)^3dw$$

$$+ \frac{8\pi^2}{\sqrt{n}}\sum_{\alpha,\beta=1}^{r}\sum_{k_1\in J_\alpha}\sum_{k_2\in J_\beta}\int_{-\pi}^{\pi}\tilde{\mathfrak{f}}_{k_1k_1}(w)^2dw\int_{-\pi}^{\pi}\tilde{\mathfrak{f}}_{k_2k_2}(w)^2dw$$

$$\times\{a_{j\alpha}a_{m\beta}a_{s\alpha\beta} + a_{j\alpha}a_{m\beta}a_{s\beta\alpha} + a_{j\alpha}a_{s\beta}a_{m\alpha\beta} + a_{j\alpha}a_{s\beta}a_{m\beta\alpha}$$

$$+ a_{s\alpha}a_{m\beta}a_{j\alpha\beta} + a_{s\alpha}a_{m\beta}a_{j\beta\alpha}\} + O(n^{-3/2})$$

$$= \frac{1}{\sqrt{n}}C_{jms} + O(n^{-3/2}),\quad (\text{say}). \qquad (3.8)$$

Therefore we get the following asymptotic expansion (see Taniguchi (1986)).

Theorem 3. Under Assumption 5, we have

$$P(L_1 < y_1,\ldots,L_k < y_k)$$

$$= \int_{-\infty}^{y_1}\cdots\int_{-\infty}^{y_k}N(\underset{\sim}{y}:\Omega)[1 + \sum_{j=1}^{k}\frac{C_j}{\sqrt{n}}H_j(\underset{\sim}{y})$$

$$+ \sum_{j,m,s=1}^{k}\frac{C_{jms}}{6\sqrt{n}}H_{jms}(\underset{\sim}{y})]d\underset{\sim}{y} + O(n^{-1}), \qquad (3.9)$$

where $\underset{\sim}{y} = (y_1,\ldots,y_k)'$, $N(\underset{\sim}{y}:\Omega) = (2\pi)^{-k/2}|\Omega|^{-1/2} \times \exp(-\frac{1}{2}\underset{\sim}{y}'\Omega^{-1}\underset{\sim}{y})$,

$$H_{j_1\ldots j_s}(\underset{\sim}{y}) = \frac{(-1)^s}{N(\underset{\sim}{y}:\Omega)} \frac{\partial^s}{\partial y_{j_1}\ldots\partial y_{j_s}} N(\underset{\sim}{y}:\Omega),$$

and $\Omega = \{c_{jm}\}$, (kxk-matrix). □

Corollary 3.1. Suppose that the eigenvalues of $\Gamma(0)$ satisfy $\lambda_1 > \ldots \lambda_p > 0$ and $T_j(\lambda_1,\ldots,\lambda_p) = \lambda_j$. In the special case when the spectral densities are constants such that

$$f_{jj}(w) = \frac{\lambda_j}{2\pi}, \quad (j=1,\ldots,p),$$

(i.e., the usual multivariate analysis case), the expansion (3.9) becomes

$$P\{\sqrt{n}(\ell_1-\lambda_1) < y_1,\ldots,\sqrt{n}(\ell_p-\lambda_p) < y_p\}$$

$$= \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_p} N(\underset{\sim}{y}:\tilde{\Omega})[1 + \sum_{j=1}^{p} \frac{\tilde{c}_j}{\sqrt{n}} H_j(\underset{\sim}{y})$$

$$+ \sum_{j,m,s=1}^{p} \frac{\ddot{c}_{jms}}{6\sqrt{n}} H_{jms}(\underset{\sim}{y})]d\underset{\sim}{y} + O(n^{-1}), \qquad (3.10)$$

where

$$\tilde{c}_j = \sum_{\beta\neq j} (\lambda_j-\lambda_\beta)^{-1}\lambda_j\lambda_\beta,$$

and the (j,m) element of $\tilde{\Omega}$ is

$$\tilde{c}_{jm} = 2\delta(j,m)\lambda_j^2,$$

and

$$\tilde{c}_{jms} = 8\lambda_j^3\delta(j,m)\delta(m,s),$$

which coincides with the result of Sugiura (1976). □

For testing $H : \lambda_{p-q} > \lambda_{p-q+1} = \ldots = \lambda_p = \theta\ (> 0)$ against $A : \lambda_{p-q+1} \geq \ldots \geq \lambda_p > 0$, we consider the criterion L defined in (2.5). Then we get the asymptotic expansion of L under the nonnull hypothesis.

Corollary 3.2. Let $T_1(\ell_{m+1},\ldots,\ell_p) = \frac{1}{n} L$ in Theorem 3 and $m = p-q$. Then, under the alternative A, we have

$$P[\sqrt{n}\{T_1(\ell_{m+1},\ldots,\ell_p) - T_1(\lambda_{m+1},\ldots,\lambda_p)\} \le y_1]$$

$$= \int_{-\infty}^{y_1} N(y_1:\Omega)\,[1 + \frac{c_1}{\sqrt{n}} H_1(y_1) + \frac{c_{111}}{6\sqrt{n}} H_{111}(y_1)]dy_1 + O(n^{-1}),$$

where c_1, Ω and c_{111} are defined in (3.4), (3.5) and (3.8), respectively, with

$$a_{1\alpha} = \frac{\sum_{i=m+1}^{p} (\lambda_\alpha - \lambda_i)}{\lambda_\alpha \sum_{i=m+1}^{p} \lambda_i},$$

$$a_{1\alpha\alpha} = \frac{1}{\lambda_\alpha^2} - \frac{p-m}{(\sum_{i=m+1}^{p} \lambda_i)^2}, \quad \alpha = m+1,\ldots,p,$$

and

$$\alpha_{1\alpha\beta} = -\frac{p-m}{(\sum_{i=m+1}^{p} \lambda_i)^2}, \quad \alpha \ne \beta,\ \alpha,\beta = m+1,\ldots,p. \qquad \square$$

4. ASYMPTOTIC EXPANSIONS OF CERTAIN FUNCTIONS OF CANONICAL CORRELATIONS

In this section we derive the asymptotic expansions of certain functions of sample canonical correlations in multivariate time series. Let $\underset{\sim}{X}(t)' = (\underset{\sim}{X}_1(t)', \underset{\sim}{X}_2(t)') = (X_1(t),\ldots,X_p(t),X_{p+1}(t),\ldots,X_{p+q}(t))$, $(p\leq q)$, be a $(p+q)$-vector-valued Gaussian stationary process with zero mean and covariance matrices $\Gamma(j) = E\{\underset{\sim}{X}(t)\underset{\sim}{X}(t+j)'\}$, which satisfy Assumption 5. Also we assume that $\{\underset{\sim}{X}(t)\}$ has the spectral density matrix $f(w) = \{f_{\alpha\beta}(w)\}$. Put

$$C(0) = \frac{1}{n}\sum_{t=1}^{n} \underset{\sim}{X}(t)\underset{\sim}{X}(t)' = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

$$\Gamma(0) = E\{C(0)\} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

$$Y = \sqrt{n}\,\{C(0)-\Gamma(0)\} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

Define the $p\times q$ matrix G as

$$G = M_{11}^{-\frac{1}{2}}\, M_{12}\, M_{22}^{-\frac{1}{2}}\,.$$

By the singular value decomposition theorem, there exist two orthogonal matrices Γ_1 and Γ_2 of order $p\times p$ and $q\times q$, respectively, such that

$$G = \Gamma_1 P \Gamma_2'$$

where $P = \{\mathrm{diag}(\rho_1,\ldots,\rho_p)|0\}$, $PP' = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, and $\lambda_i = \rho_i^2$, $(i = 1,\ldots,p)$. Define

$$\pi_1 = \Gamma_1' M_{11}^{-\frac{1}{2}}\,,$$

$$\pi_2 = \Gamma_2' M_{22}^{-\frac{1}{2}}\,.$$

Then, using a similar argument to Fang and Krishnaiah (1982), we can see

$$(\pi_1')^{-1}S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}\pi_1'$$

$$= PP' + \frac{1}{\sqrt{n}}(PV_{21} - PV_{22}P' + V_{12}P' - V_{11}PP')$$

$$+ \frac{1}{n}(PV_{22}V_{22}P' - PV_{22}V_{21} + V_{12}V_{21} - V_{12}V_{22}P'$$

$$- V_{11}PV_{21} + V_{11}PV_{22}P' - V_{11}V_{12}P' + V_{11}V_{11}PP')$$

$$+ 0_p(n^{-3/2}), \tag{4.1}$$

where $V_{ij} = \pi_i Y_{ij}\pi_j'$, $(i,j = 1,2)$.

Now, without loss of generality we assume that $\{\underset{\sim}{X}(t)\}$ has the covariance matrix

$$\Gamma(0) = \begin{pmatrix} 1 & 0 & \rho_1 & 0 & \\ & \ddots & & \ddots & 0 \\ 0 & 1 & 0 & \rho_p & \\ \rho_1 & 0 & 1 & 0 & \\ 0 & \rho_p & 0 & 1 & \\ & 0 & & & \end{pmatrix}, \tag{4.2}$$

$$\tilde{\delta}(w) = \{\tilde{\delta}_{jk}(w)\} = \begin{pmatrix} \delta_{11}^{(1)}(w) & 0 & \delta_{12}^{(1)}(w) & 0 & \\ & \ddots & & \ddots & 0 \\ 0 & \delta_{11}^{(p)}(w) & 0 & \delta_{12}^{(p)}(w) & \\ \delta_{21}^{(1)}(w) & 0 & \delta_{22}^{(1)}(w) & 0 & \\ 0 & \delta_{21}^{(p)}(w) & 0 & \delta_{22}^{(q)}(w) & \\ & 0 & & & \end{pmatrix} \tag{4.3}$$

with

$$\rho_j = \int_{-\pi}^{\pi} f_{21}^{(j)}(w)dw = \int_{-\pi}^{\pi} f_{12}^{(j)}(w)dw, \qquad j = 1,\ldots,p,$$

$$1 = \int_{-\pi}^{\pi} f_{11}^{(j)}(w)dw, \ j = 1,\ldots,p,$$

$$1 = \int_{-\pi}^{\pi} f_{22}^{(j)}(w)dw, \ j = 1,\ldots,q.$$

So we may assume that $\pi_1 = I_p$ and $\pi_2 = I_q$. Let $\ell_1 \geq \cdots \geq \ell_p$ be the eigenvalues of $S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}$, and let $T_j(\ell_1,\ldots,\ell_p)$, $(j = 1,\ldots,k)$, be an analytic function with the same notations defined in the previous section. We set down

$$L_j = \sqrt{n}\,\{T_j(\ell_1,\ldots,\ell_p) - T_j(\lambda_1,\ldots,\lambda_p)\}\ ;\ j = 1,\ldots,k,$$

where $\lambda_1,\ldots,\lambda_p$ satisfy the relation (3.2). Then, using the same arguments and notations we have

$$L_j = \sum_{\alpha=1}^{r} a_{j\alpha}\ \mathrm{tr}\ W_\alpha^{(1)} + \frac{1}{\sqrt{n}}\{\sum_{\alpha=1}^{r} a_{j\alpha}\ \mathrm{tr}\ W_\alpha^{(2)}$$

$$+ \tfrac{1}{2}\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r} a_{j\alpha\beta}\ (\mathrm{tr}W_\alpha^{(1)})(\mathrm{tr}W_\beta^{(1)}) \ + \text{higher order terms}, \qquad (4.4)$$

with

$$W_\alpha^{(1)} = U_{\alpha\alpha}^{(1)},$$

$$W_\alpha^{(2)} = U_{\alpha\alpha}^{(2)} + \sum_{\beta\neq\alpha}(\theta_\alpha-\theta_\beta)^{-1}U_{\alpha\beta}^{(1)}U_{\beta\alpha}^{(1)},$$

where

$$U^{(1)} = PY_{21} - PY_{22}P' + Y_{12}P' - Y_{11}PP',$$

$$U^{(2)} = PY_{22}Y_{22}P' - PY_{22}Y_{21} + Y_{12}Y_{21}$$

$$- Y_{12}Y_{22}P' - Y_{11}PY_{21} + Y_{11}PY_{22}P'$$

$$- Y_{11}Y_{12}P' + Y_{11}Y_{11}PP'.$$

Define

$$U(k,m) = \rho_k Y(p+k,m) - \rho_k \rho_m Y(p+k,p+m)$$

$$+ \rho_m Y(k,p+m) - \rho_m^2 Y(k,m),$$

$$V(m,k) = \rho_k Y(k+p,m+p) - Y(k,m+p),$$

where $Y(\alpha,\beta)$ is the (α,β)th component of the matrix Y. Then, by (4.4) we have

$$L_j = \sum_{\alpha=1}^{r} a_{j\alpha} \{ \sum_{k\varepsilon J_\alpha} U(k,k)\}$$

$$+ \frac{1}{\sqrt{n}} \{ \sum_{\alpha=1}^{r} a_{j\alpha} [\sum_{i=1}^{r+1} \sum_{k\varepsilon J_\alpha} \sum_{m\varepsilon J_i} V(m,k)^2$$

$$- \sum_{i=1}^{r} \sum_{m\varepsilon J_\alpha} \sum_{m\varepsilon J_i} Y(k,m)U(m,k)$$

$$+ \sum_{\beta\neq\alpha} (\theta_\alpha - \theta_\beta)^{-1} \sum_{k\varepsilon J_\alpha} \sum_{m\varepsilon J_\beta} U(k,m)U(m,k)]$$

$$+ \tfrac{1}{2} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} a_{j\alpha\beta} \sum_{k\varepsilon J_\alpha} \sum_{m\varepsilon J_\beta} U(k,k)U(m,m)\}$$

$$+ \text{higher order terms,} \tag{4.5}$$

where J_{r+1} denotes the set of integers $p+1,\ldots,q$. We denote

$$K_{vv}(m,k:m,k) = E\{V(m,k)\}^2,$$

$$K_{yu}(k,m:m,k) = E\{Y(k,m)U(m,k)\},$$

$$K_{uu}(k,m:m,k) = E\{U(k,m)U(m,k)\},$$

$$K_{uv}(\ell,\ell:m,k) = E\{U(\ell,\ell)V(m,k)\}.$$

$$K_{uuu}(k:m:s) = \sqrt{n}\ \text{cum}\{U(k,k),U(m,m),U(s,s)\}.$$

Then using Lemma 1 for Gaussian case we can see that

$$K_{vv}(m,k:m,k)$$

$$= 2\pi\rho_k^2 \int_{-\pi}^{\pi}\{1+\delta(m,k)\}f_{22}^{(k)}(w)f_{22}^{(m)}(w)dw$$

$$- 4\pi\rho_k \int_{-\pi}^{\pi}\{f_{21}^{(k)}(w)f_{22}^{(m)}(w) + \delta(m,k)f_{22}^{(k)}(w)f_{12}^{(m)}(w)\}dw$$

$$+ 2\pi \int_{-\pi}^{\pi}\{f_{11}^{(k)}(w)f_{22}^{(m)}(w) + \delta(m,k)f_{12}^{(k)}(w)^2\}dw + O(n^{-1}), \tag{4.6}$$

$$K_{yu}(k,m:\ell,\ell) = [4\pi\rho_\ell \int_{-\pi}^{\pi} f_{11}^{(\ell)}(w)\{f_{12}^{(\ell)}(w) + f_{21}^{(\ell)}(w)\}dw$$

$$- 4\pi\rho_\ell^2 \int_{-\pi}^{\pi} f_{12}^{(\ell)}(w)f_{21}^{(\ell)}(w)dw - 4\pi\rho_\ell^2 \int_{-\pi}^{\pi} f_{11}^{(\ell)}(w)^2dw]\delta(k,\ell)\delta(m,\ell)$$

$$+ O(n^{-1}), \tag{4.7}$$

$$
\begin{aligned}
& K_{uu}(k,m:m,k) \\
&= 2\pi\rho_k^2\rho_m^2 \int_{-\pi}^{\pi} \{1+\delta(m,k)\}\{f_{11}^{(k)}(w) f_{11}^{(m)}(w) + f_{22}^{(k)}(w) f_{22}^{(m)}(w)\}dw \\
&- 2\pi\rho_k\rho_m^2 \int_{-\pi}^{\pi} [\{f_{12}^{(k)}(w) + \delta(m,k) f_{21}^{(k)}(w)\} f_{11}^{(m)}(w) \\
&+ \{1+\delta(m,k)\} f_{12}^{(k)}(w) f_{22}^{(m)}(w) + \{f_{21}^{(k)}(w) + \delta(m,k) f_{12}^{(k)}(w)\} f_{22}^{(m)}(w)]dw \\
&- 2\pi\rho_k^2\rho_m \int_{-\pi}^{\pi} [\{1+\delta(m,k)\}\{f_{11}^{(k)}(w) f_{12}^{(m)}(w) + f_{22}^{(k)}(w) f_{21}^{(m)}(w)\} \\
&+ \{f_{12}^{(m)}(w) + \delta(m,k) f_{21}^{(m)}(w)\} f_{22}^{(k)}(w)]dw \\
&-2\pi\rho_k^3 \int_{-\pi}^{\pi} \{1+\delta(m,k)\} f_{21}^{(k)}(w) f_{11}^{(m)}(w)dw \\
&-2\pi\rho_m^3 \int_{-\pi}^{\pi} \{f_{21}^{(m)}(w) + \delta(m,k) f_{12}^{(m)}(w)\} f_{11}^{(k)}(w)dw \\
&+2\pi\rho_k^2 \int_{-\pi}^{\pi} \{f_{22}^{(k)}(w) f_{11}^{(m)}(w) + \delta(m,k) f_{21}^{(k)}(w) f_{21}^{(m)}(w)\}dw \\
&+2\pi\rho_m^2 \int_{-\pi}^{\pi} \{f_{22}^{(m)}(w) f_{11}^{(k)}(w) + \delta(m,k) f_{12}^{(k)}(w) f_{12}^{(m)}(w)\}dw \\
&+2\pi\rho_k\rho_m \int_{-\pi}^{\pi} \{f_{21}^{(k)}(w) f_{21}^{(m)}(w) + \delta(m,k) f_{22}^{(k)}(w) f_{11}^{(m)}(w) \\
&\qquad + f_{12}^{(k)}(w) f_{12}^{(m)}(w) + \delta(m,k) f_{11}^{(k)}(w) f_{22}^{(m)}(w)\}dw \\
&+2\pi\rho_k^3\rho_m \int_{-\pi}^{\pi} f_{21}^{(k)}(w) f_{12}^{(m)}(w)\{1+\delta(m,k)\}dw \\
&+2\pi\rho_k\rho_m^3 \int_{-\pi}^{\pi} f_{12}^{(k)}(w) f_{21}^{(m)}(w)\{1+\delta(m,k)\}dw + O(n^{-1}), \qquad (4.8)
\end{aligned}
$$

$$K_{uu}(m,k:\ell,\ell) = \begin{cases} O(n^{-1}) & \text{for all} \quad m \neq k,\ \ell \\ O(n^{-1}) & \text{for all} \quad m = k \neq \ell, \end{cases}$$

$$K_{uv}(\ell,\ell:m,k) = 10\pi\rho_\ell^2 \int_{-\pi}^{\pi} f_{22}^{(\ell)}(w) f_{21}^{(\ell)}(w)dw$$

$$-4\pi\rho_\ell^3 \int_{-\pi}^{\pi}\{f_{22}^{(\ell)}(w)^2 + f_{12}^{(\ell)}(w)f_{21}^{(\ell)}(w)\}dw$$

$$-4\pi\rho_\ell \int_{-\pi}^{\pi}\{f_{21}^{(\ell)}(w)^2 + f_{22}^{(\ell)}(w)f_{11}^{(\ell)}(w)\}dw$$

$$+2\pi\rho_\ell^2 \int_{-\pi}^{\pi}\{f_{22}^{(\ell)}(w)f_{12}^{(\ell)}(w) + (f_{21}^{(\ell)}(w) + f_{12}^{(\ell)}(w))f_{11}^{(\ell)}(w)\}dw$$

$$+ O(n^{-1}), \qquad \text{for } \ell = m = k,$$

$$= O(n^{-1}), \qquad \text{otherwise.} \tag{4.9}$$

Let Δ be the set of integers k and p+k, and let

$$\gamma(j_1,j_2) = \begin{cases} 2\rho_k, & \text{if } |j_1-j_2| = 1, \\ -\rho_k^2, & \text{if } |j_1-j_2| = 0. \end{cases}$$

Then, using Lemma 3 we have

$$K_{uuu}(k:m:s) = (2\pi)^2 \sum_{j_1,\ldots,j_6\in\Delta} \gamma(j_1,j_2)\gamma(j_3,j_4)\gamma(j_5,j_6)$$

$$\times \sum_{\nu}^{*} \int_{-\pi}^{\pi} \tilde{f}_{\nu_1\nu_2}(w)\tilde{f}_{\nu_3\nu_4}(w)\tilde{f}_{\nu_5\nu_6}(w)dw + O(n^{-1}),$$

$$\text{for } k = m = s,$$

$$= 0 \quad , \qquad \text{otherwise,} \tag{4.10}$$

where $\sum_{\nu}^{*}$ is the sum of all two dimensional indecomposable partions of $\left\{\begin{matrix}(j_1,j_2)\\(j_3,j_4)\\(j_5,j_6)\end{matrix}\right\}$.

Thus noting (4.5)-(4.10), it is easy to show that

$$E(L_j) = \frac{1}{\sqrt{n}} C_j^{(1)} + O(n^{-3/2}), \tag{4.11}$$

$$\mathrm{cum}\{L_j, L_s\} = C_{js}^{(2)} + O(n^{-1}), \tag{4.12}$$

$$\mathrm{cum}\{L_j, L_s, L_\ell\} = \frac{1}{\sqrt{n}} C_{js\ell}^{(3)} + O(n^{-3/2}), \tag{4.13}$$

where

$$C_j^{(1)} = \sum_{\alpha=1}^{r} a_{j\alpha} \{ \sum_{i=1}^{r+1} \sum_{k\varepsilon J_\alpha} \sum_{m\varepsilon J_i} K_{vv}(m,k:m,k)$$

$$- \sum_{i=1}^{r} \sum_{k\varepsilon J_\alpha} \sum_{m\varepsilon J_i} K_{yu}(k,m:m,k)$$

$$+ \sum_{\beta\neq\alpha} (\theta_\alpha - \theta_\beta)^{-1} \sum_{k\varepsilon J_\alpha} \sum_{m\varepsilon J_\beta} K_{uu}(k,m:m,k)\}$$

$$+ \tfrac{1}{2} \sum_{\alpha=1}^{r} a_{j\alpha\alpha} \sum_{k\varepsilon J_\alpha} K_{uu}(k,k:k,k), \tag{4.14}$$

$$C_{js}^{(2)} = \sum_{\alpha=1}^{r} a_{j\alpha} a_{s\alpha} \sum_{k\varepsilon J_\alpha} K_{uu}(k,k:k,k), \tag{4.15}$$

$$C_{js\ell}^{(3)} = \sum_{\alpha=1}^{r} a_{j\alpha} a_{s\alpha} a_{\ell\alpha} \sum_{k\varepsilon J_\alpha} \{K_{uuu}(k:k:k)$$

$$+ 6K_{uv}(k,k:k,k)^2 - 6K_{uy}(k,k:k,k)K_{uu}(k,k:k,k)\}$$

$$+ \sum_{\alpha,\beta=1}^{r} \{a_{j\alpha}a_{s\beta}a_{\ell\alpha\beta} + a_{j\beta}a_{\ell\alpha}a_{s\alpha\beta} + a_{\ell\beta}a_{s\alpha}a_{j\alpha\beta}\}$$

$$\times \sum_{k\in J_\alpha} \sum_{m\in J_\beta} K_{uu}(k,k:k,k)K_{uu}(m,m:m,m). \tag{4.16}$$

Therefore we get the following asymptotic expansion (see Taniguchi (1986)).

Theorem 4

$$P(L_1 < y_1,\dots,L_k < y_k)$$

$$= \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_k} N(\underset{\sim}{y}:\Omega)[1 + \sum_{j=1}^{k} \frac{C_j^{(1)}}{\sqrt{n}} H_j(\underset{\sim}{y})$$

$$+ \sum_{j,s,\ell=1}^{k} \frac{C_{js\ell}^{(3)}}{6\sqrt{n}} H_{js\ell}(\underset{\sim}{y})]d\underset{\sim}{y} + O(n^{-1}), \tag{4.17}$$

where

$$\Omega = \{C_{js}^{(2)}\}, \text{ (k×k-matrix)}. \qquad \square$$

Now consider the test of

$$H : \rho_k^2 > \rho_{k+1}^2 = \dots = \rho_p^2 = 0, \tag{4.18}$$

with

$$f_{12}^{(k+1)}(w) = \dots = f_{12}^{(p)}(w) = 0,$$

and

$$f_{11}^{(k+1)}(w) = \dots = f_{11}^{(p)}(w) = f_{22}^{(k+1)}(w) = \dots = f_{22}^{(q)}(w) = \frac{1}{2\pi},$$

against

$$A : \rho_{k+1}^2 \geq \cdots \geq \rho_p^2 \geq 0 \text{ and } \rho_{k+1}^2 > 0. \tag{4.19}$$

For this test problem we use the following statistic

$$Q = -n\log \prod_{j=k+1}^{p} (1-\ell_j). \tag{4.20}$$

In the usual multivariate analysis (4.20) is known as the likelihood ratio statistic for testing H. Then, under H, it is not difficult to show

$$Q = \sum_{\ell=k+1}^{p} \sum_{j=k+1}^{q} Y(\ell,p+j)^2 + \text{higher order terms}. \tag{4.21}$$

Thus we have

<u>Proposition 1</u>. When the null hypothesis H is true, the limiting distribution as $n \to \infty$ of Q is the χ^2-distribution with $(p-k)(q-k)$ degrees of freedom. □

Using Theorem 4 we can get the asymptotic expansion of Q under the non-null hypothesis.

<u>Proposition 2</u>. Let $T_1(\ell_{k+1},\ldots,\ell_p) = \frac{1}{n} Q$ in Theorem 4. Then, under A, we have

$$P[\sqrt{n}\{T_1(\ell_{k+1},\ldots,\ell_p) - T_1(\rho_{k+1}^2,\ldots,\rho_p^2)\} < y_1]$$

$$= \int_{-\infty}^{y_1} N(y_1;\Omega)[1 + \frac{C_1^{(1)}}{\sqrt{n}} H_1(y_1) + \frac{C_{111}^{(3)}}{6\sqrt{n}} H_{111}(y_1)]dy_1$$

$$+ O(n^{-1}),$$

where $C_1^{(1)}$, Ω and $C_{111}^{(3)}$ are defined in (4.14), (4.15) and (4.16), with

$$a_{1\alpha} = \frac{1}{1 - \rho_\alpha^2},$$

$$a_{1\alpha\alpha} = \frac{1}{(1 - \rho_\alpha^2)^2}, \quad \alpha = k+1, \ldots, p,$$

and

$$a_{1\alpha\beta} = 0 \ , \ \alpha \neq \beta. \qquad \square$$

5 APPLICATIONS IN SIGNAL PROCESSING

Consider the following model in the area of signal processing.

$$\underset{\sim}{x}(t) = A\underset{\sim}{s}(t) + \underset{\sim}{n}(t) \tag{5.1}$$

where $A = [A(\underset{\sim}{\Phi}_1),\ldots,A(\underset{\sim}{\Phi}_q)]$, $\underset{\sim}{s}(t) = (s_1(t),\ldots,s_q(t))'$ $\underset{\sim}{n}(t) = (n_1(t),\ldots,n_p(t)]'$ and $q< p$. Here $\underset{\sim}{n}(t)$ is complex Gaussian white noise process with $E(\underset{\sim}{n}(t)) = 0$, $E(\underset{\sim}{n}(t)\overline{\underset{\sim}{n}(t)}') = \sigma^2 I$ covariance matrix $\sigma^2 I_p$. Also, $s_i(t)$ is the wave form associated with i-th signal. In addition $\underset{\sim}{s}(t)$ is assumed dimentional complex Gaussian stationary to be process with $E(\underset{\sim}{s}(t)) = \underset{\sim}{0}$ and $E(\underset{\sim}{s}(t)\overline{\underset{\sim}{s}(t)}') = \psi$. The processes $\underset{\sim}{s}(t)$ and $\underset{\sim}{n}(t)$ are independent of each other. Wax and Kailath (1984) and Zhao, Krishnaiah and Bai (1985) considered the problem of determination of the number of unknown signals by using information theoretic criteria under certain assumptions. The number of signals is equal to the $q = p-m$ where m is the multiplicity of the smallest eigenvalue of the covariance matrix Σ of $\underset{\sim}{x}(t)$. We assume that $\underset{\sim}{x}(t)$ is a complex stationary Gaussian process and let $nS = \sum_{j=1}^{n} \underset{\sim}{x}(t_j)\overline{\underset{\sim}{x}(t_j)}'$. Test of hypotheses for the multiplicity of the smallest eigenvalue of Σ are based upon certain functions of the eigenvalues of S and so the results in this paper are useful in determination of the number of signals.

Next, let

$$\underset{\sim}{y}(t) = B\underset{\sim}{s}(t) + \underset{\sim}{n}_0(t) \tag{5.2}$$

where $\underset{\sim}{n}_0(t)$ is defined as $\underset{\sim}{n}(t)$, $B{:}p\times q$ is defined as A and $\underset{\sim}{y}(t)$ is the observation vector on p sensors located at different locations. In this case, let us assume that $(\underset{\sim}{x}'(t),\underset{\sim}{y}'(t))$ is distributed as complex, stationary, Gaussian process. The results in Section 4 are useful for testing for independence of $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ processes.

REFERENCES

Anderson, T.W. (1984). An Introduction to Multivariate Statistical Analysis. Second Edition. Wiley, New York.

Brillinger, D.R. (1975). Time Series: Data Analysis and Theory. Holden Day, New York.

Fang, C. and Krishnaiah, P.R. (1982). Asymptotic distributions of functions of the eigenvalues of some random matrices for nonnormal populations. J. Multivariate Anal. 12, 39-63.

Fang, C. and Krishnaiah, P.R. (1982). Asymptotic joint distributions of functions of the elements of sample covariance matrix. In: G. Kallianpur, P.R. Krishnaiah, J.K. Ghosh, eds., Statistics and Probability Essays in Honor of C.R. Rao. North-Holland Publishing Company.

Fujikoshi, Y. (1977). Asymptotic expansions for the distributions of some multivariate tests. In: P.R. Krishnaiah, ed., Mulitivariate Analysis IV. North-Holland Publishing Company

Fujikoshi, Y. (1978). Asymptotic expansions for the distributions of some functions of the latent roots of matrices in three situations. J. Multivariate Anal. 8, 63-72.

Hannan, E.J. (1970). Multiple Time Series. Wiley, New York.

Hosoya, Y. and Taniguchi, M. (1982). A central limit theorem for stationary processes and the parameter estimation of linear processes. Ann. Stat. 10, 132-153.

Krishnaiah, P.R. (1976). Some recent developments on complex multivariate distributions. J. Multivariate Anal., 6, 1-30.

Krishnaiah, P.R. and Lee, J.C. (1979). On the asymptotic joint distributions of certain functions of the eigenvalues of four random matrices. J. Multivariate Anal. 9, 248-258.

Lawley, D.N. (1959). Test of significance in canonical analysis. Biometrika, 46, 59-66.

Murihead, R.J. and Waternaux, C.M. (1980). Asymptotic distributions in canonical correlation analysis and other multivariate procedures for nonnormal populations. Biometrika, 67, 31-43.

Priestly, M.B., Rao, T. Subba, and Tong, H. (1973). Identification of the structure of multivariate stochastic systems. In: Multivariate Analysis - III, P.R. Krishnaiah, Ed. New York, Academic Press, 351-368.

Robinson, P.M. (1973). Generalized canonical analysis for time series. J. Multivariate Anal. 3, 141-160.

Sugiura, N. (1976). Asymptotic expansions of the distributions of the latent roots and the latent vector of the Wishart and multivariate F matrices. J. Multivariate Anal. 6, 500-525.

Taniguchi, M. (1983). On the second order asymptotic efficiency of estimators of Gaussian ARMA processes. Ann. Stat. 11, 157-169.

Taniguchi, M. (1986). Third order asymptotic properties of maximum likelihood estimators for Gaussian ARMA processes. To appear in J. Multivariate Anal. 18, 1-31.

Zhao, L.C., Krishnaiah, P.R. and Bai, Z.D. (1985). On detection of number of signals in presence of white noise. Technical Report No. 85-37. Center for Multivariate Analysis, University of Pittsburgh.